SIGN ANALYSIS TECHNIQUE

The following are obviously true when graphing any function \( y = f(x) \).

- When \( f(x) \) is positive, the curve will be above the x-axis.
- When \( f(x) \) is negative, the curve will be below the x-axis.
- When \( f(x) = 0 \), the curve will intersect the x-axis.

Consider the graph of some function \( y = f(x) \) as shown below.

![Graph of \( y = f(x) \)](image)

Note that \( f(x) \) is positive for all values of \( x \) below 1. This could be expressed in interval notation as \( f(x) \) is positive on \((-\infty, 1)\).

Using similar notation, we could describe \( f(x) \) for all values of \( x \).

- \( f(x) \) is positive on the interval \((-\infty, 1)\).
- \( f(x) \) is positive on the interval \((1, 3)\).
- \( f(x) \) is negative on the interval \((3, 4)\).
- \( f(x) \) is positive on the interval \((4, 6]\).
- \( f(x) \) is negative on the interval \((6, \infty)\).

Note that \( f(x) \) changes sign (changes from negative to positive or positive to negative) at \( x = 3 \), \( x = 4 \), and \( x = 6 \). Note also that \( f(3) = 0 \), \( f(4) \) is undefined, and function \( f \) is discontinuous at \( x = 6 \). ("Discontinuous" means that you cannot graph the function without lifting your pencil. Expect a more precise definition in future math courses.)

We can make the following general conclusion.

\[
\begin{array}{c}
\text{f(x) can change sign only for special values of x such that one of the following is true:} \\
f(x) = 0 \\
f(x) \text{ is undefined} \\
f \text{ is discontinuous at x}
\end{array}
\]

The special values of \( f \) are \( \{1, 3, 4, 6\} \). The sign of \( f(x) \) changes at 3, 4, and 6. The number 1 is a special value, but \( f(x) \) does not change sign at \( x = 1 \).
Using Sign Analysis for Graphing Functions

Example: Graph \( y = f(x) = x^4 - x^3 - 8x^2 + 12x \)

Factoring this polynomial gives \( y = f(x) = x(x - 2)^2(x + 3) \). The function is defined and continuous for all values of \( x \). Since \( f(0) = f(2) = f(-3) = 0 \), the special values are \( \{0, 2, -3\} \).

These special values divide the domain of function \( f \) into four intervals as shown below.

\[
\begin{array}{cccc}
 & I & II & III & IV \\
\hline
-3 & & & & \\
0 & & & & \\
2 & & & & \\
\end{array}
\]

Since \( f(x) \) can only change signs at these three special values, we know that in any given interval, \( f(x) \) is either always positive or always negative. We can therefore test just one value of \( x \) in each interval.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>( f(x) )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>-4</td>
<td>144 (positive)</td>
<td>The graph is above the X-axis.</td>
</tr>
<tr>
<td>II</td>
<td>-1</td>
<td>-18 (negative)</td>
<td>The graph is below the X-axis.</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>4 (positive)</td>
<td>The graph is above the X-axis.</td>
</tr>
<tr>
<td>IV</td>
<td>3</td>
<td>18 (positive)</td>
<td>The graph is above the X-axis.</td>
</tr>
</tbody>
</table>

* Note that it is not necessary to actually compute values for \( f(x) \). It is only necessary to decide whether \( f(x) \) is positive or negative.

From the sign analysis, we can make a rough sketch of the given polynomial.

Using Sign Analysis for Solving Inequalities

Example: Solve for \( x \) if \( x^4 - x^3 - 8x^2 + 12x > 0 \).

Note that this is the polynomial \( f(x) \) from the previous example.

Factoring \( f(x) \) gives \( x(x - 2)^2(x + 3) > 0 \). We can now use the table shown above and note that \( f(x) > 0 \) on intervals I, III, and IV. Therefore, the given inequality is true for all values of \( x \) such that \( x < -3 \) or \( 0 < x < 2 \) or \( x > 2 \).

Example: Solve for \( x \) if \( x^4 - x^3 - 8x^2 + 12x \geq 0 \).

The table shown above can be used once again, except this time the equality is allowed. Therefore, the inequality is true for all values of \( x \) such that \( x \leq -3 \) or \( x \geq 0 \).

Summary of Sign Analysis Technique

1. Begin by finding all special values of the polynomial.
2. Locate the special values on the X-axis. This will divide the domain into intervals.
3. Select a value of \( x \) from each interval and compute \( f(x) \).
4. Recognize that if \( f(x) \) is positive for one value in an interval, then \( f(x) \) is positive for all values in the interval. If \( f(x) \) is negative for one value in the interval, then \( f(x) \) is negative for all values in the interval.
5. Consider \( f(x) \) at the special values (interval endpoints) and answer the question.